Screen bundles of Lorentzian manifolds and some generalisations of pp-waves

Thomas Leistner*

School of Mathematical Sciences, The University of Adelaide, SA 5005, Australia

Abstract

A pp-wave is a Lorentzian manifold admitting a parallel light-like vector field and satisfying a certain curvature condition. On the basis of a new description of pp-waves we introduce generalisations of pp-waves, on one hand by allowing the vector field to be recurrent and on the other hand by weakening the curvature condition. We show how these generalisations can be expressed in terms of the screen bundle of the manifold and how they are related to the screen holonomy. While pp-waves have a trivial screen holonomy there are no restrictions on the screen holonomy of the generalised pp-waves defined by the weaker curvature condition.
© 2005 Elsevier B.V. All rights reserved.

1. Introduction

Regarding holonomy theory or the existence of parallel spinors, undoubtedly the most interesting Lorentzian manifolds are those with indecomposable, but non-irreducible holonomy representation. This is due to the de Rham/Wu decomposition theorem ([12] and [25]) which says that any complete semi-Riemannian manifold is isometric to a product of complete semi-Riemannian manifolds with indecomposable holonomy representation. This implies that a

---

* Tel.: +61(0)8 83033712; fax: +61(0)8 83033696.
  E-mail address: tleistne@maths.adelaide.edu.au.

Lorentzian manifold is isometric to a product of irreducible or flat Riemannian manifolds and a Lorentzian manifold, the holonomy representation of which is either trivial, irreducible or indecomposable. Now, due to Berger's classification result [6], all irreducible holonomy algebras are known and the corresponding geometric structures are studied extensively. But in the de Rham/Wu decomposition there may occur a Lorentzian factor with indecomposable but non-irreducible holonomy representation. Hence it remains to study those manifolds. Following the fundamental work by Berard-Bergery and Ikemakhen in [5] recently the holonomy groups of such manifolds have been classified in [17,19,20] and [14], but very little is known about the corresponding geometric structures.

The first thing to observe is that a Lorentzian manifold with indecomposable, non-irreducible holonomy representation admits a recurrent light-like vector field and its holonomy algebra is contained in the parabolic algebra $(\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$, provided that the dimension of the manifold is $n + 2$. The main ingredient of this holonomy algebra is its $\mathfrak{so}(n)$-projection, which is called *screen holonomy*. In order to solve the classification problem, we addressed ourselves in [17,19] and [20] to the classification of the screen holonomy and obtained the result that it has to be a Riemannian holonomy.

On the other hand it can be shown that any Riemannian holonomy group can be realised as screen holonomy of an indecomposable, non-irreducible Lorentzian manifold by a rather simple method: for a Riemannian manifold (N, g) and $f \in C^\infty(N)$ the manifold $\mathbb{R}^2 \times N$ with Lorentzian metric $2\mathrm{d}x\mathrm{d}z + f\mathrm{d}z^2 + g$ is non-irreducible, indecomposable for f sufficiently generic and, above all, its screen holonomy is equal to the Riemannian holonomy of (N, g).

In this note we want to consider Lorentzian manifolds which are in some sense complementary to the ones obtained by this procedure and which can be understood as certain generalisations of pp-waves. pp-waves are defined by the existence of a light-like parallel vector field and a certain curvature condition. pp-waves are important in several respects. They play an important role in string theory where they occur as Penrose limits of supersymmetric $AdS \times S$ solutions (see for example [8]). A certain subclass of pp-waves, the so-called *Cahen–Wallach spaces*, provides all non-irreducible, indecomposable Lorentzian symmetric spaces ([11], see also [5]). Another subclass is given by *plane waves* which represent the gravitational waves in General Relativity (see for example [23] for details).

Our aim is to generalise pp-waves in two directions: on one hand we will only require the existence of a recurrent vector field instead of a parallel one (see Section 4), and on the other hand, more importantly, we will relax the curvature condition (see Section 5). The second generalisation is based on a description of pp-waves which we have obtained in [22]. Both generalisations are related to the screen holonomy in the following sense. In Section 3 we introduce the so-called *screen bundle* of a Lorentzian manifold with recurrent light-like vector field and prove that the screen holonomy can be understood as holonomy of this screen bundle. Now, pp-waves have trivial screen holonomy, i.e. their screen bundle is flat. This remains true if we drop the assumption that the vector field is parallel, but it is no longer true if we weaken the curvature condition. Instead, we prove in Section 5 that the screen bundles restricted to the light-like hypersurfaces defined by the recurrent vector field are flat.

These generalisations can also be understood in terms of the local form of a Lorentzian metric h with recurrent light-like vector field and some parts of the paper will deal with this local form. The ingredients of this form are a function f, a family of one-forms ϕ_z and a family of Riemannian metrics g_z since h can be written as $h = 2\mathrm{d}x\mathrm{d}z + f\mathrm{d}z^2 + \phi\mathrm{d}z + g_z$. For a pp-wave we have $\phi = 0$, g_z flat and not depending on z and $\frac{\partial}{\partial x}(f) = 0$. If we no longer require a parallel vector field only the conditions $\phi = 0$ and g_z flat and not depending on z remain.

Finally, weakening the curvature conditions is equivalent to dropping the assumption $\phi = 0$, i.e. only requiring g_z to be flat and not depending on z. As mentioned this is complementary to the construction method above where $\phi = 0$ was assumed.

Although the defining curvature conditions on these generalised pp-waves are only slightly weaker than for pp-waves the consequences for the screen holonomy are dramatic in the following sense. While for pp-waves the screen holonomy has to be trivial, any possible screen holonomy, that is any Riemannian holonomy, can be obtained for the generalisations of pp-waves. This can be deduced from a recent result in [14] and is explained in the last section.

2. Lorentzian manifolds with recurrent light-like vector field

A vector field X on a semi-Riemannian manifold (M, h) with Levi-Civita connection ∇ is called *recurrent* if $\nabla X = \Theta \otimes X$ where Θ is a one-form on M. If the length of a recurrent vector field is non-zero, it can be rescaled to a parallel one. This is not true in general if the recurrent vector field is light-like.

If an $(n + 2)$-dimensional Lorentzian manifold (M, h) carries a recurrent light-like vector field X the holonomy group of (M, h) in $p \in M$ admits the one-dimensional light-like invariant subspace $\mathbb{R} \cdot X_p$ and therefore does not act irreducibly. The orthogonal complement $X_p^{\perp}$ of this subspace is $(n + 1)$-dimensional, holonomy invariant as well and contains $\mathbb{R} \cdot X_p$, i.e. there is a holonomy invariant flag

$$\mathbb{R} \cdot X_p \subset \mathbb{R} \cdot X_p^{\perp} \subset T_p M.$$

Hence, X yields two parallel distributions, a one-dimensional, totally isotropic distribution Ξ with $X \in \Gamma(\Xi)$, and its $(n + 1)$-dimensional orthogonal complement

$$\Xi^{\perp} = \{U \in TM \mid h(U, X) = 0\}$$

containing Ξ. Both foliate the manifold into light-like lines $\mathcal{X}$, which are the flow of X, and light-like hypersurfaces $\mathcal{X}^{\perp}$. Using this foliation the following coordinate description was proven ([24, 10] and [23]).

Proposition 1. *Let (M, h) be a Lorentzian manifold of dimension $n + 2 > 2$ with recurrent light-like vector field X.*

1. *This is equivalent to the existence of coordinates $(U, \varphi = (x, (y_i)_{i=1}^n, z))$ in which the metric h has the following local shape*

$$h = 2\mathrm{d}x\mathrm{d}z + \sum_{i=1}^{n} u_i \mathrm{d}y_i \mathrm{d}z + f\mathrm{d}z^2 + \sum_{i,j=1}^{n} g_{ij}\mathrm{d}y_i\mathrm{d}y_j \tag{1}$$

   with $\frac{\partial g_{ij}}{\partial x} = \frac{\partial u_i}{\partial x} = 0$, $f \in C^{\infty}(M)$. We refer to these coordinates as Walker coordinates.
2. *X is parallel if and only if f does not depend on x. We refer to these coordinates as Brinkmann coordinates.*
3. *If X is parallel the coordinates can be chosen such that $u_i = 0$ and even that $f = 0$. We refer to these coordinates as Schimming coordinates.*

A Lorentzian manifold with light-like parallel vector field is called a *Brinkmann wave*, after [10]. For further coordinate descriptions see [9] or [21].

Returning to the holonomy group of a Lorentzian manifold with recurrent light-like vector field we want to mention some of its algebraic properties. The holonomy algebra $\mathfrak{h}$ of an $(n+2)$-dimensional Lorentzian manifold with recurrent light-like vector field is a subalgebra of the parabolic algebra $\mathfrak{p} = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$ which is given in an appropriate basis as

$$\mathfrak{p} = \left\{ \begin{pmatrix} a & v^t & 0 \\ 0 & A & -v \\ 0 & 0 & -a \end{pmatrix} \,\middle|\, a \in \mathbb{R}, v \in \mathbb{R}^n, A \in \mathfrak{so}(n) \right\}.$$

Its projection onto $\mathbb{R}^n$ is surjective if and only if the holonomy group acts indecomposably. The recurrent vector field is parallel if and only if the holonomy is contained in $\mathfrak{so}(n) \ltimes \mathbb{R}^n$. There are four different algebraic types of holonomy algebras, two of them *uncoupled*, i.e. $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^n$ and $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^n$, and two with a coupling between the center of the $\mathfrak{so}(n)$-projection and the $\mathbb{R}$—resp. the $\mathbb{R}^n$—part (for details see [5]). Further algebraic properties can be proven easily.

Lemma 1. *Let $\mathfrak{h}$ be an indecomposable subalgebra of the parabolic algebra. Then:*

1. $\mathfrak{h}$ *is solvable if and only if it is 2-step solvable (i.e. $\mathfrak{h}^{(1)} \neq 0$ and $\mathfrak{h}^{(2)} = 0$) or Abelian.*
2. $\mathfrak{h}$ *is Abelian if and only if $\mathfrak{h} = \mathbb{R}^n$.*
3. *If $\mathfrak{h}$ is the holonomy algebra of an indecomposable, non-irreducible Lorentzian manifold, then $\mathfrak{h}$ is 2-step solvable if and only if $\mathfrak{h} = \mathbb{R} \ltimes \mathbb{R}^n$ or the screen holonomy algebra equals a direct sum of copies of $\mathfrak{so}(2)$.*

Proof. The first point is obvious from the commutator relations in the parabolic algebra

$$[(a, A, x), (b, B, y)] = (0, [A, B], (A + a\,Id)y - (B + b\,Id)x).$$

Set $\mathfrak{g} := pr_{\mathfrak{so}(n)}\mathfrak{h}$. If $\mathfrak{h}$ is solvable, then $\mathfrak{g}$ has to be solvable. But, as a subalgebra of $\mathfrak{so}(n)$, $\mathfrak{g}$ is reductive, i.e. it is solvable if and only if it is Abelian. Hence $\mathfrak{h}^{(1)} \subset \mathfrak{g}^{(1)} \ltimes \mathbb{R}^n = \mathbb{R}^n$ and therefore $\mathfrak{h}^{(2)} = 0$. From the commutator relation one sees that $\mathfrak{h}^{(1)} = 0$ only if $\mathfrak{h} = \mathbb{R}^n$. The remaining decomposition of the $\mathfrak{so}(n)$—part $\mathfrak{g}$ under the assumption that $\mathfrak{h}$ is a holonomy algebra follows from a Borel–Lichnerowicz decomposition theorem proved in [5]. $\square$

3. The screen bundle associated to a recurrent light-like vector field

In this section we will describe the $SO(n)$-projection of an indecomposable, non-irreducible holonomy group of an $(n+2)$-dimensional, simply connected Lorentzian manifold as a holonomy group of a metric connection in a vector bundle, the so called *screen bundle*. For more details see [21].

We consider the distributions Ξ and $\Xi^\perp$ on M introduced in Section 2, which are parallel, i.e. ∇_U leaves $\Gamma(\Xi)$ and $\Gamma(\Xi^\perp)$ invariant for all $U \in TM$. The quotients $\Xi_p^\perp / \Xi_p$ in every point $p \in M$ define a vector bundle over M,

$$\mathcal{S} := \bigcup_{p \in M}^{\cdot} \Xi_p^\perp / \Xi_p,$$

which is called *screen bundle*. The metric h on M defines a scalar product on $\mathcal{S}$, which we denote by $\hat{h}$, via

$$\hat{h}([X], [Y]) := h(X, Y).$$

With respect to this scalar product the bundle $\mathcal{O}(\mathcal{S})$ is defined as the set of orthonormal frames of $\mathcal{S}$ over M. This is a $O(n)$-principal fibre bundle. $\mathcal{O}(\mathcal{S})$ has fibres

$$\mathcal{O}_p(\mathcal{S}) = \left\{ ([E_1], \ldots, [E_n]) \,\middle|\, \begin{array}{l} (X, E_1, \ldots, E_n) \text{ a basis of } \Xi_p^\perp \text{ for } X \in \Xi_p \\ \text{with } h(E_i, E_j) = \delta_{ij} \end{array} \right\}.$$

Then we can describe $\mathcal{S}$ as a vector bundle associated with the bundle $\mathcal{O}(\mathcal{S})$:

$$\mathcal{O}(\mathcal{S}) \times_{O(n)} \mathbb{R}^n \simeq \mathcal{S}$$

$$[([E_1], \ldots, [E_n]), (x_1, \ldots x_n)] \mapsto \left[\sum_{i=1}^n x_i E_i \right].$$

Now we consider the subbundle $\mathcal{P}(M, h)$ of the frame bundle with fibres

$$\mathcal{P}_p(M, h) := \left\{ (X, E_1, \ldots, E_n, Z) \,\middle|\, \begin{array}{l} X \in \Xi_p, E_i \in \Xi_p^\perp, h(E_i, E_j) = \delta_{ij}, \\ h(Z, Z) = h(Z, E_i) = 0, h(X, Z) = 1 \end{array} \right\} \tag{2}$$

and structure group $P = (\mathbb{R}^* \times O(n)) \ltimes \mathbb{R}^n$. We define a surjective bundle homomorphism

$$\begin{array}{rccc} f : & \mathcal{P}(M, h) & \to & \mathcal{O}(\mathcal{S}) \\ & (X, E_1, \ldots, E_n, Z) & \mapsto & ([E_1], \ldots, [E_n]). \end{array}$$

Then f defines a reduction of the projection $pr_{O(n)} : P = (\mathbb{R}^* \times O(n)) \ltimes \mathbb{R}^n \to O(n)$.

Lemma 2. $f : \mathcal{P}(M, h) \to \mathcal{O}(\mathcal{S})$ *is a* $pr_{O(n)}$-*reduction.*

Proof. We have to verify that the following diagram commutes

$$\begin{array}{ccc} P \times \mathcal{P}(M, h) & \longrightarrow & \mathcal{P}(M, h) \\ pr_{SO(n)} \times f \downarrow \quad \circlearrowleft \quad f \downarrow \quad \circlearrowleft & & \searrow \\ & & M. \\ O(n) \times \mathcal{O}(\mathcal{S}) & \longrightarrow & \mathcal{O}(\mathcal{S}) \quad \nearrow \end{array}$$

The action of the components of P on $\mathcal{P}(M, h)$ is as follows:

$$(X, E_1, \ldots, E_n, Z) \cdot (a, Id, 0) = (aX, E_1, \ldots, E_n, a^{-1}Z) \tag{3}$$

and

$$\begin{aligned} &(X, E_1, \ldots, E_n, Z) \cdot (1, Id, v) \\ &= \left(X, v_1 X + E_1, \ldots, v_n X + E_n, -\frac{1}{2} v^t v X - \sum_{k=1}^n v_k E_k + Z \right). \end{aligned} \tag{4}$$

Since P is a semi-direct product this implies that

$$f((X, E_1, \ldots, E_n, Z) \cdot (a, A, v)) = ([E_1], \ldots, [E_n]) \cdot A.$$

But this makes the above diagram commutative. $\quad \square$

Since Ξ is parallel the Levi-Civita connection defines a covariant derivative $\nabla^{\mathcal{S}}$ on $\mathcal{S}$ by

$$\nabla_X^{\mathcal{S}}[Y] := [\nabla_X Y].$$

This covariant derivative is metric with respect to $\hat{h}$ since the Levi-Civita connection is metric. It defines a connection form θ on $\mathcal{O}(\mathcal{S})$ which is given for a local section $\hat{\sigma} = ([\sigma_1], \ldots, [\sigma_n]) \in \Gamma(\mathcal{O}(\mathcal{S}))$ by the formula

$$\nabla_U^{\mathcal{S}}[V] = \nabla_U^{\mathcal{S}}[(\hat{\sigma}, v)] = [\hat{\sigma}, \mathrm{d}v(V) + \theta^{\hat{\sigma}}(U) \cdot v]$$

for $v = (v_1, \ldots, v_n)$ and $[V] = \sum_{i=1}^n v_i[\sigma_i]$ locally, where $\theta^{\hat{\sigma}}$ is the local connection form of θ. We get the following result.

Proposition 2. *Let (M, h) be a Lorentzian manifold of dimension $n + 2$ and with parallel isotropic distribution Ξ. Let ω denote the connection form of the Levi-Civita connection ∇. Then ω is a $pr_{O(n)}$-reduction of the connection θ of $\mathcal{O}(\mathcal{S})$.*

Proof. We consider the diagram

$$
\begin{array}{ccc}
T\mathcal{P}(M, h) & \xrightarrow{\ df\ } & T\mathcal{O}(\mathcal{S}) \\
\omega \downarrow & & \downarrow \theta \\
\mathfrak{p} & \xrightarrow[dpr_{O(n)} = pr_{\mathfrak{so}(n)}]{} & \mathfrak{so}(n)
\end{array}
\tag{5}
$$

and have to show that $(df)_s$ sends the kernel of ω_s to the kernel of $\theta_{f(s)}$ for $s \in \mathcal{P}(M, h)$.

Every element in the kernel of ω_s is equal to $(d\sigma)_p(U)$ for $p \in M$, $U \in T_p M$ and a certain local section $\sigma \in \Gamma(\mathcal{P}(M, h))$ with $\sigma(p) = s$. Now we have

$$0 = \omega_{\sigma(p)}((d\sigma)_p(U)) = (\sigma^* \omega)_p(U) = \omega_p^{\sigma}(U).$$

For the local connection form ω^{σ} of the Levi-Civita connection one calculates as follows: for $\sigma = (\xi, \sigma_1, \ldots, \sigma_n, \zeta) \in \Gamma(\mathcal{P}(M, h))$ and E_{rt} the standard basis of $\mathfrak{gl}(n+2, \mathbb{R})$ we have

$$
\begin{aligned}
0 = \omega^{\sigma}(U) &= h(\nabla_U \xi, \zeta)(E_{00} - E_{n+1n+1}) \qquad \text{(the } \mathbb{R}\text{-part)} \\
&\quad + \sum_{k=1}^n h(\nabla_U \sigma_k, \zeta)(E_{0k} - E_{kn+1}) \qquad (\text{ the } \mathbb{R}^n\text{-part)} \\
&\quad + \sum_{1 \leq k < l \leq n} h(\nabla_U \sigma_k, \sigma_l)(E_{kl} - E_{lk}) \qquad \text{(the } \mathfrak{so}(n)\text{-part)}.
\end{aligned}
$$

We have to consider $(df)_{\sigma(p)}(d\sigma)_p(U) = d(f \circ \sigma)_p(U)$. If now $\sigma \in \Gamma(\mathcal{P}(M, h))$ as above, then $f \circ \sigma = ([\sigma_1], \ldots, [\sigma_n]) \in \Gamma(\mathcal{O}(\mathcal{S}))$. Finally we have

$$
\begin{aligned}
\theta_{f \circ \sigma(p)}(d(f \circ \sigma)_p(U)) &= \theta^{f \circ \sigma}(U) \\
&= \sum_{1 \leq k < l \leq n} \hat{h}(\nabla_U^{\mathcal{S}}[\sigma_k], [\sigma_l])(E_{kl} - E_{lk}) \\
&= \sum_{1 \leq k < l \leq n} h(\nabla_U \sigma_k, \sigma_l)(E_{kl} - E_{lk}) \\
&= 0
\end{aligned}
$$

because of the equation above. That is, $d(f \circ \sigma)_p(U)$ is in the kernel of the local connection $\theta^{f \circ \sigma}$. Hence it is in the kernel of θ. $\quad\square$

Corollary 1. *The diagram (5) commutes and the curvatures Θ of θ and Ω of ω satisfy*

$$f^* \Theta = pr_{\mathfrak{so}(n)} \circ \Omega.$$

In particular, the holonomy algebras are related in the following way,

$$\mathfrak{hol}_p(\mathcal{S}, \nabla^{\mathcal{S}}) = pr_{\mathfrak{so}(n)}(\mathfrak{hol}_p(M, h)).$$

Proof. This follows from the proposition by the general theory of reductions of connections. Since f and df are surjective one gets by the Ambrose–Singer holonomy theorem that $\mathfrak{hol}_{f(s)}(\theta) = pr_{\mathfrak{so}(n)}(\mathfrak{hol}_s(\omega))$. $\square$

In [17,19] and [20] we have shown that the screen holonomy $\mathfrak{g} := pr_{\mathfrak{so}(n)}(\mathfrak{hol}(M, h))$ has to be a Riemannian holonomy algebra. Furthermore, the description of $\mathfrak{g}$ as holonomy of the screen bundle can be used to interpret the geometric information which is algebraically encoded in $\mathfrak{g}$ as geometric structure on the screen bundle $\mathcal{S}$. For example, if there is a complex structure on $\mathcal{S}$ which is compatible with the metric $\hat{h}$ and parallel to the covariant derivative $\nabla^{\mathcal{S}}$ then the flag $\varXi \subset \varXi^{\perp} \subset TM$ is called the *Kähler flag*. The existence of such a Kähler flag is equivalent to $\mathfrak{g} \subset \mathfrak{u}(n)$. For $\mathfrak{g} \subset \mathfrak{su}(n)$ one calls such a flag a *special Kähler flag*. For details see [3] and [16]. This can be done analogously for any other geometric structure on $\mathcal{S}$, or algebraic structure on $\mathfrak{g}$.

4. Lorentzian manifolds with trivial screen holonomy

In this section we want to collect some results about pp-waves which lead to a further generalisation of pp-waves in the next section. But first we recall the definition of a pp-wave. A Brinkmann wave is called a *pp-wave* if its curvature tensor $\mathcal{R}$ satisfies the trace condition $tr_{(3,5)(4,6)}(\mathcal{R} \otimes \mathcal{R}) = 0$. Schimming proved the following coordinate description and equivalences in [23].

Lemma 3. *A Lorentzian manifold (M, h) of dimension $n + 2 > 2$ is a pp-wave if and only if there exist local coordinates $(U, \varphi = (x, (y_i)_{i=1}^n, z))$ in which the metric h has the form*

$$h = 2\mathrm{d}x\mathrm{d}z + f\mathrm{d}z^2 + \sum_{i=1}^{n} \mathrm{d}y_i^2, \qquad with \ \frac{\partial f}{\partial x} = 0. \tag{6}$$

Lemma 4. *A Brinkmann wave (M, h) with parallel light-like vector field X is a pp-wave if and only if one of the following conditions — in which ξ denotes the 1-form $h(X, .)$ — is satisfied:*

1. *$\Lambda_{(1,2,3)}(\xi \otimes \mathcal{R}) = 0$.*
2. *There is a symmetric $(2, 0)$-tensor ϱ with $\varrho(X, .) = 0$, such that $\mathcal{R} = \Lambda_{(1,2)(3,4)}(\xi \otimes \varrho \otimes \xi)$.*
3. *There is a function φ, such that $tr_{(1,5)(4,8)}(\mathcal{R} \otimes \mathcal{R}) = \varphi \xi \otimes \xi \otimes \xi \otimes \xi$.*

In [22] we gave another equivalence for the definition which seems to be simpler than any of the trace conditions and which allows a generalisation in the next section.

Proposition 3. *A Brinkmann wave (M, h) with parallel light-like vector field X and induced parallel distributions $\varXi$ and $\varXi^{\perp}$ is a pp-wave if and only if its curvature tensor satisfies*

$$\mathcal{R}(U, V) : \varXi^{\perp} \longrightarrow \varXi \qquad for \ all \ U, V \in TM, \tag{7}$$

or equivalently

$$\mathcal{R}(Y_1, Y_2) = 0 \qquad for \ all \ Y_1, Y_2 \in \varXi^{\perp}. \tag{8}$$

From this description one obtains easily that a pp-wave is Ricci-isotropic and has vanishing scalar curvature. But it also enables us to introduce a first generalisation of pp-waves by supposing (7) but only the existence of a recurrent light-like vector field. Assuming that the abbreviation 'pp' stands for 'plane fronted with parallel rays' we call them *pr-waves*, 'plane fronted with recurrent rays'.

Definition 1. We call a Lorentzian manifold (M, h) a *pr-wave* if it admits a recurrent light-like vector field X and its curvature tensor $\mathcal{R}$ obeys

$$\mathcal{R}(U, V) : \Xi^\perp \longrightarrow \Xi \qquad \text{for all } U, V \in TM, \tag{9}$$

or equivalently $\mathcal{R}(Y_1, Y_2) = 0$ for all $Y_1, Y_2 \in X^\perp$.

Since X is not parallel all the trace conditions which were true for a pp-wave fail to hold for a pr-wave. For example, if we suppose (9) we get for the trace $tr_{(3,5)(4,6)}(\mathcal{R} \otimes \mathcal{R})(U, V, W, Z) = h_p(\mathcal{R}(U, V)X, \mathcal{R}(W, Z)Z)$ which is not necessarily zero. But one can prove an equivalence similar to 1 of Lemma 4. We proved the following lemmata and propositions in [22].

Lemma 5. *A Lorentzian manifold (M, h) with recurrent light-like vector field X is a pr-wave if and only if $\Lambda_{(1,2,3)}(\xi \otimes \mathcal{R}) = 0$, where ξ denotes the 1-form $h(X, .)$.*

Also we get a description in terms of local coordinates similar to that for pp-waves.

Lemma 6. *A Lorentzian manifold (M, h) of dimension $n + 2 > 2$ is a pr-wave if and only if around any point $o \in M$ there exist coordinates $(U, \varphi = (x, (y_i)_{i=1}^n, z))$ in which the metric h has the following form,*

$$h = 2dxdz + f dz^2 + \sum_{i=1}^n dy_i^2, \qquad \text{with } f \in C^\infty(M). \tag{10}$$

Regarding the vanishing of the screen holonomy the following result can be obtained by the description of Proposition 3 and the definition of a pr-wave.

Proposition 4. *A Lorentzian manifold (M, h) with recurrent light-like vector field is a pr-wave if and only if the following equivalent conditions hold:*

(1) *The screen holonomy of (M, h) is trivial (i.e. the screen bundle over M is flat).*
(2) *(M, h) has solvable holonomy contained in $\mathbb{R} \ltimes \mathbb{R}^n$.*

In addition, (M, h) is a pp-wave if and only if its holonomy is Abelian, i.e. contained in $\mathbb{R}^n$.

Finally, we have proven that the property of being Ricci-isotropic forces a pr-wave to be a pp-wave.

Proposition 5. *A pr-wave is a pp-wave if and only if it is Ricci-isotropic.*

For sake of completeness we also want to mention two subclasses of pp-waves. The first are the *plane waves* which are pp-waves with quasi-recurrent curvature, i.e. $\nabla \mathcal{R} = \xi \otimes \tilde{\mathcal{R}}$ where $\xi = h(X, .)$ and $\tilde{\mathcal{R}}$ a $(4, 0)$-tensor. For plane waves the function f in the local form of the metric is of the form $f = \sum_{i,j=1}^n a_{ij} y_i y_j$ where the a_{ij} are functions of z. A subclass of plane waves are the Lorentzian symmetric spaces with solvable transvection group, the so-called *Cahen–Wallach spaces* (see [11], also [5]). For these the function f satisfies $f = \sum_{i,j=1}^n a_{ij} y_i y_j$ where the a_{ij} are constants.

5. Another generalisation of pp-waves

Now we want to introduce another class of Lorentzian manifolds by relaxing also the curvature condition.

Definition 2. We say that a Lorentzian manifold (M, h) with recurrent light-like vector field has *light-like hypersurface curvature* if its curvature tensor $\mathcal{R}$ obeys

$$\mathcal{R}(U, V) : \Xi^\perp \longrightarrow \Xi \qquad \text{for all } U, V \in \Xi^\perp, \tag{11}$$

where Ξ and $\Xi^\perp$ are the light-like distributions defined by the recurrent vector field.

Of course, (11) is equivalent to the fact that the (4, 0)-curvature tensor vanishes on $\Xi^\perp \times \Xi^\perp \times \Xi^\perp \times \Xi^\perp$.

The chosen name can be explained by the following considerations. Since (M, h) carries a recurrent light-like vector field, the manifold is foliated into the flow of this vector field and the submanifolds defined by the integrable distribution $\Xi^\perp$. Hence, through any point $p \in M$ goes a one-dimensional isotropic submanifold $\mathcal{X}_p$ and a light-like hypersurface $\mathcal{X}_p^\perp$ with tangent bundles $T\mathcal{X}_p = \Xi|_{\mathcal{X}_p}$ and $T\mathcal{X}_p^\perp = \Xi^\perp|_{\mathcal{X}_p^\perp}$ respectively, satisfying $\mathcal{X}_p \subset \mathcal{X}_p^\perp$. Since the distribution $\Xi^\perp$ is parallel, i.e. $\nabla_U : \Gamma(\Xi^{(\perp)}) \to \Gamma(\Xi^{(\perp)})$ for every $U \in TM$, the Levi-Civita connection ∇ of (M, h) defines a connection on the hypersurface $\mathcal{X}_p^\perp$, denoted by $\overset{\circ}{\nabla} : \Gamma(T\mathcal{X}_p^\perp \otimes T\mathcal{X}_p^\perp) \to \Gamma(T\mathcal{X}_p^\perp)$. Then we get the following equivalences.

Proposition 6. *A Lorentzian manifold with recurrent light-like vector field X has light-like hypersurface curvature if and only if every light-like hypersurface $\mathcal{X}_p^\perp$, defined by X and equipped with induced connection $\overset{\circ}{\nabla}$, satisfies one of the following equivalent conditions:*

(1) *If $\overset{\circ}{\mathcal{R}}$ is the curvature of $\overset{\circ}{\nabla}$, then for any $U, V, W \in T\mathcal{X}_p^\perp$ the tangent vector $\overset{\circ}{\mathcal{R}}(U, V)W$ is light-like.*

(2) *The holonomy of $\overset{\circ}{\nabla}$ is solvable and contained in $\mathbb{R} \ltimes \mathbb{R}^n$.*

If in addition X is parallel, then the holonomy of $\overset{\circ}{\nabla}$ is Abelian and contained in $\mathbb{R}^n$.

Proof. First we prove the equivalence of the two conditions under the assumption that (M, h) is a Lorentzian manifold with recurrent light-like vector field X. The equivalence is based on the Ambrose–Singer holonomy theorem which says that $\mathfrak{hol}_q(\mathcal{X}_p^\perp, \overset{\circ}{\nabla})$ is generated by the following endomorphisms of $T_q\mathcal{X}_p^\perp = \Xi_q^\perp$,

$$\overset{\circ}{\mathcal{P}}_\gamma^{-1} \circ \overset{\circ}{\mathcal{R}}(U, V) \circ \overset{\circ}{\mathcal{P}}_\gamma \in \mathfrak{gl}(T_q\mathcal{X}_p^\perp),$$

where $\overset{\circ}{\mathcal{P}}_\gamma$ is the parallel displacement w.r.t. $\overset{\circ}{\nabla}$ along a curve γ in $\mathcal{X}_p^\perp$ starting at q, and $U, V \in \Xi_{\gamma(1)}^\perp$. For γ being the constant curve it becomes evident that (2) implies (1). Now bearing in mind that $\nabla = \overset{\circ}{\nabla}$ on $\mathcal{X}_p$ which implies that $\overset{\circ}{\mathcal{P}}_\gamma$ leaves Ξ invariant, (1) implies that the holonomy algebra maps $T_q\mathcal{X}_p^\perp = \Xi_q^\perp$ onto Ξ_q which means it is contained in

$$\left\{ \begin{pmatrix} a & v^t \\ 0 & 0 \end{pmatrix} \,\middle|\, a \in \mathbb{R}, v \in \mathbb{R}^n \right\} \subset \mathfrak{gl}(n+1)$$

with respect to a basis adapted to $\Xi_q \subset \Xi_q^\perp$. In addition, when X is parallel it is mapped to zero by the holonomy algebra as $\overset{\circ}{\mathcal{R}}(U, V)X = 0$. Finally it is evident that the condition (11) from the definition is equivalent to (1). $\quad\square$

In [7] the quantities assigned to the hypersurfaces $\mathcal{X}_p^\perp$ are used to describe the holonomy of a Lorentzian manifold further, in particular to decide to which type in the distinction given in [5] by Berard-Bergery and Ikemakhen the holonomy algebra belongs. This approach makes use of a *screen distribution* which is complementary and orthogonal to Ξ in $\Xi^\perp$ (see also [4], or recently [1] and [13]). Such a screen distribution can always be chosen, but since it requires a choice we prefer to work with an analogon to the screen bundle introduced in Section 3 which can be defined without making such a choice.

Let $\mathcal{X}_p^\perp$ be a light-like hypersurface through $p \in M$ defined by the recurrent light-like vector field X. Then we define the *restricted screen bundle* over $\mathcal{X}_P^\perp$ as

$$\overset{\circ}{\mathcal{S}} := \mathcal{S}|_{\mathcal{X}_p^\perp}.$$

$\overset{\circ}{\mathcal{S}}$ is equipped with a covariant derivative defined by $\overset{\circ}{\nabla}$,

$$\nabla_U^{\overset{\circ}{\mathcal{S}}}[V] := [\overset{\circ}{\nabla}_U V], \qquad \text{for } U \in T\mathcal{X}_p^\perp, V \in \Gamma(T\mathcal{X}_p^\perp).$$

Again, since Ξ is $\overset{\circ}{\nabla}$-invariant, this is well defined. We obtain another equivalence in terms of the screen bundle.

Proposition 7. *A Lorentzian manifold with recurrent light-like vector field X has light-like hypersurface curvature if and only if over every light-like hypersurface $\mathcal{X}_p^\perp$ defined by X the connection $\nabla^{\overset{\circ}{\mathcal{S}}}$ on the restricted screen bundle $\overset{\circ}{\mathcal{S}}$ is flat.*

Proof. The curvature $\mathcal{R}^{\overset{\circ}{\mathcal{S}}}$ of $\nabla^{\overset{\circ}{\mathcal{S}}}$ can be written in terms of the curvature of $\overset{\circ}{\nabla}$ as

$$\mathcal{R}^{\overset{\circ}{\mathcal{S}}}(U, V)[W] = [\overset{\circ}{\mathcal{R}}(U, V)W], \qquad \text{for } U, V, W \in T\mathcal{X}_p^\perp.$$

Then the previous proposition gives the equivalence. $\quad\square$

For the case where the vector field X is parallel we obtain the following equivalent trace condition.

Proposition 8. *A Brinkmann wave (M, h) has light-like hypersurface curvature if and only if the curvature tensor $\mathcal{R}$ of (M, h) obeys $\|\mathcal{R}\|^2 = 0$ where $\|\mathcal{R}\|^2$ is the square of the norm of the curvature tensor, defined by $\|\mathcal{R}\|^2 := tr_{(1,5)(2,6)(3,7)(4,8)}(\mathcal{R} \otimes \mathcal{R})$.*

Proof. Again we use the basis $(X, E_1, \ldots, E_n, Z)$ as in (2). Because X is parallel every curvature term where X is plugged in vanishes and we get

$$\|\mathcal{R}\|^2 = \sum_{i,j,k,l=1}^n \mathcal{R}(E_i, E_j, E_k, E_l)^2.$$

But this expression vanishes if and only if (M, h) satisfies (11). $\quad\square$

Now we want to focus on the description of a Lorentzian manifold with light-like hypersurface curvature in local coordinates.

Proposition 9. *A Lorentzian manifold (M, h) of dimension $n+2 > 2$ has light-like hypersurface curvature if and only if around any point $o \in M$ there exist coordinates $(U, \varphi = (x, (y_i)_{i=1}^n, z))$ in which the metric h has the following local shape*

$$h = 2\mathrm{d}x\mathrm{d}z + f\mathrm{d}z^2 + \left(\sum_{i=1}^n u_i \mathrm{d}y_i \right) \mathrm{d}z + \sum_{i=1}^n \mathrm{d}y_i^2 \tag{12}$$

with $\frac{\partial u_i}{\partial x} = 0$ and $f \in C^\infty(M)$. If, in addition, (M, h) is a Brinkmann wave, then f does not depend on x. In the corresponding Schimming coordinates $(u_i = 0)$ the g_{ij} are the coefficients of a z-dependent family of flat Riemannian metrics.

Proof. For Walker coordinates $(x, y_1, \ldots, y_n, z)$ the condition that $\mathcal{R}$ vanishes on $\Xi^\perp$ gives that $\mathcal{R}(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l}) = 0$. But this is the integrability condition for the existence of new coordinates with $h(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}) = \delta_{ij}$. This proves the first point.

If we now chose Schimming coordinates we still have the condition $\mathcal{R}(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l}) = 0$. But for Schimming coordinates $\mathcal{R}(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l})$ is equal to $\mathcal{R}^g(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l})$ where $\mathcal{R}^g$ denotes the curvature tensor of the Riemannian metrics defined by the coefficients g_{ij}. Hence for each z this has to be a flat Riemannian metric. $\square$

The description in these coordinates shows that the $\mathfrak{so}(n)$-part of the curvature and the holonomy is generated by expressions of the form $\mathcal{R}(\frac{\partial}{\partial z}, \frac{\partial}{\partial y_i})$ as we will see in the following. We will illustrate this description in two different types of coordinates by some calculations.

First we calculate the curvature of a Lorentzian manifold with light-like hypersurface curvature in a point and given coordinates of the form (12). We can arrange these coordinates around the point p in such a way that $\frac{\partial}{\partial x}, (\frac{\partial}{\partial y_i})_{i=1}^n, \frac{\partial}{\partial z}$ is a basis of the form $(X, E_1, \ldots, E_n, Z)$ as in (2). Hence, if $\mathfrak{g} := pr_{\mathfrak{so}(n)}(\mathfrak{hol}_p(M, h))$, then $\mathfrak{g}$ contains the following elements of $\mathfrak{so}(n)$, for each $U, V \in T_p M$:

$$\sum_{i,j=1}^n h\left(\mathcal{R}(U, V)\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right) E_{ij}$$

where E_{ij} denotes the standard basis of $\mathfrak{so}(n)$. Now the only non-vanishing curvature terms of this form are

$$h\left(\mathcal{R}\left(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial z} \right) \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right) = \mathcal{R}\left(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial z}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right)$$
$$= \frac{1}{2}\frac{\partial}{\partial y_k}\left(\frac{\partial}{\partial y_i}(u_j) - \frac{\partial}{\partial y_j}(u_i) \right).$$

Hence, if one finds functions $(u_1, \ldots, u_n)$ with $\frac{\partial}{\partial y_k}\left(\frac{\partial}{\partial y_i}(u_j) - \frac{\partial}{\partial y_j}(u_i) \right) \neq 0$ one obtains a non-irreducible, non-indecomposable Lorentzian manifold with light-like curved hypersurfaces, but with non-trivial screen holonomy.

Now we calculate the curvature of such a manifold in Schimming coordinates, i.e. with $u_i = 0$, i.e.

$$h = 2\mathrm{d}x\mathrm{d}z + f\mathrm{d}z^2 + \sum_{i,j=1}^n g_{ij}\mathrm{d}y_i\mathrm{d}y_j.$$

Having light-like hypersurface curvature implies that $\mathcal{R}(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l}) = 0$, i.e. that g_{ij} is a z-dependent family of flat Riemannian metrics. Let us denote by Γ_{ij}^k its z-dependent Christoffel symbols. Then we get the following for the only non-vanishing curvature terms which are relevant for the $\mathfrak{so}(n)$-projection of the holonomy:

$$
\mathcal{R}\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_p}, \frac{\partial}{\partial z}\right) = \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y_j}(g_{ip}) - \frac{\partial}{\partial y_i}(g_{jp})\right)
$$
$$
+ \frac{1}{2}\sum_{k=1}^{n}\left(\Gamma_{ip}^k \frac{\partial}{\partial z}(g_{jk}) - \Gamma_{jp}^k \frac{\partial}{\partial z}(g_{ik})\right).
$$

In order to construct a non-irreducible, indecomposable Lorentzian manifold with light-like curved hypersurfaces and non-trivial screen holonomy one has to find a family of flat Riemannian metrics with Christoffel symbols such that the above expression is non-zero.

Now we prove further properties of the coordinates.

Proposition 10. *A Lorentzian manifold with light-like hypersurface curvature is a pr-wave, i.e. has trivial screen holonomy, if there exist local coordinates of the form* (12) *such that the z-dependent family of one-forms $\phi := \sum_{k=1}^{n} u_k \mathrm{d}y_k$ on $\mathbb{R}^n$ is closed for any z.*

Proof. Since the ϕ_z are closed — considered as a family of differential forms on $\mathbb{R}^n$ — they are a differential of a function φ which does not depend on the x coordinate. More exactly: if $\phi_z = \sum_{k=1}^{n} u_k \mathrm{d}y_k$ with $\frac{\partial}{\partial x}(u_k) = 0$ and

$$
0 = \mathrm{d}\phi_z = \sum_{l=1}^{n} \mathrm{d}u_l \wedge \mathrm{d}y_l = \sum_{k,l=1}^{n} \frac{\partial}{\partial y_k}(u_l)\mathrm{d}y_k \wedge \mathrm{d}y_l
$$

then there exists a $\beta \in C^\infty(M)$ with $\frac{\partial}{\partial x}(\beta) = 0$ and $u_k = \frac{\partial}{\partial y_k}(\beta)$.

Now we consider the following coordinates

$$
\tilde{x} = x + \beta, \qquad \tilde{y}_i = y_i, \qquad \tilde{z} = z. \tag{13}
$$

These satisfy $\tilde{u}_i = 0$, $\tilde{g}_{ij} = \delta_{ij}$ and $\tilde{f} = f - 2\frac{\partial}{\partial z}(\beta)$, and are therefore coordinates of a pr-wave. $\square$

As regards the Ricci curvature we can prove the condition for the Ricci isotropy in terms of the form ϕ. The Ricci isotropy is an important property because it is a necessary condition for the existence of parallel spinors on (M, h) (for a proof see [2]).

Proposition 11. *A Brinkmann wave with light-like hypersurface curvature is Ricci-isotropic if and only if there are coordinates for which the family of one-forms $\phi_z := \sum_{k=1}^{n} u_k \mathrm{d}y_k$ on $\mathbb{R}^n$ satisfies the equation*

$$
\mathrm{d}^* \mathrm{d}\phi_z = 0. \tag{14}
$$

for all z.

Proof. We consider ϕ_z as a family of one-forms on $\mathbb{R}^n$. Fixing coordinates of the form (12) we get that the basis

$$
X := \frac{\partial}{\partial x}
$$

$$Z := \frac{\partial}{\partial z} - \frac{f}{2} \frac{\partial}{\partial x}$$

$$E_k := \frac{\partial}{\partial y_i} - u_i \frac{\partial}{\partial x}.$$

is of the form (2). In these coordinates and this basis we obtain as $\frac{\partial}{\partial x}$ is parallel:

$$
\begin{aligned}
Ric\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial y_i}\right) &= \overbrace{\mathcal{R}\left(\frac{\partial}{\partial z}, X, Z, \frac{\partial}{\partial y_i}\right)}^{=0} \\
&\quad + \overbrace{\mathcal{R}\left(\frac{\partial}{\partial z}, X, Z, \frac{\partial}{\partial y_i}\right)}^{=0} + \sum_{k=1}^{n} \mathcal{R}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_i}\right) \\
&= -\frac{1}{2} \sum_{k=1}^{n}\left[\frac{\partial}{\partial y_k}\left(\frac{\partial}{\partial y_k}\left(\phi_z\left(\frac{\partial}{\partial y_i}\right)\right)\right) - \frac{\partial}{\partial y_k}\left(\frac{\partial}{\partial y_i}\left(\phi_z\left(\frac{\partial}{\partial y_k}\right)\right)\right)\right] \\
&= d^*d\phi_z\left(\frac{\partial}{\partial y_i}\right).
\end{aligned}
$$

Here d^* is the co-differential with respect to the flat Riemannian metric $g \equiv \delta_{ij}$. But a Brinkmann wave is Ricci-isotropic if and only if $Ric(Y, .) = 0$ for every $Y \in \Xi^\perp$ (see for example [22] for a proof) which gives the statement. $\quad\square$

6. Further remarks on holonomy and examples

We want to start the concluding remarks about the holonomy of Lorentzian manifolds with light-like hypersurface curvature with an example.

Example 1. There are examples which show that Lorentzian manifolds with light-like hypersurface curvature can have non-trivial screen holonomy, in particular having irreducible screen holonomy $\mathfrak{so}(3) \subset \mathfrak{so}(5)$ given by the Riemannian symmetric pair. The first example of such a manifold was given in [15], although with another purpose. One considers the following one-form $\phi = \sum_{k=1}^{5} u_k dy_k$ on $\mathbb{R}^5$ with

$$u_1 = -y_3^2 - 4y_4^2 - y_5^2,$$
$$u_3 = -2\sqrt{3}y_2 y_3 - 2y_4 y_5,$$
$$u_5 = 2\sqrt{3}y_2 y_5 + 2y_3 y_4,$$
$$u_2 = u_4 = 0.$$

Now one defines the Lorentzian metric on $\mathbb{R}^7$ by

$$h := 2dxdz + fdz^2 + \phi dz + \sum_{k=1}^{5} dy_k^2$$

where f is a function on $\mathbb{R}^7$ with $\frac{\partial f}{\partial y_i} \neq 0$. The holonomy of this manifold equals $(\mathbb{R}\oplus\mathfrak{so}(3, \mathbb{R})) \ltimes \mathbb{R}^5$ or if f does not depend on x equals $\mathfrak{so}(3, \mathbb{R}) \ltimes \mathbb{R}^5$ where $\mathfrak{so}(3, \mathbb{R}) \subset \mathfrak{so}(5, \mathbb{R})$ is the irreducible representation defined by the Riemannian symmetric pair: the Lie algebra $\mathfrak{sl}(3, \mathbb{R})$

can be decomposed into vector spaces $\mathfrak{sl}(3, \mathbb{R}) = \mathfrak{so}(3, \mathbb{R}) \oplus sym_0(3, \mathbb{R})$, where $sym_0(3, \mathbb{R})$ denote the trace free symmetric matrices. This is a five-dimensional vector space, invariant and irreducible under the adjoint action of $\mathfrak{so}(3, \mathbb{R})$. This representation is equal to the holonomy representation of the Riemannian symmetric space $Sl(3, \mathbb{R})/SO(3, \mathbb{R})$.

Another example of this type having the same holonomy was constructed in [21] by setting $u_1 = -4y_1y_2$, $u_2 = 4y_1y_2$, $u_3 = -y_1y_4 - y_2y_4 + y_1y_3 - y_2y_3 + \sqrt{3}(y_4y_5 - y_3y_5)$, $u_4 = y_1y_4 - y_2y_4 + y_1y_3 + y_2y_3 + \sqrt{3}(y_4y_5 + y_3y_5)$ and $u_5 = 0$. Recently in [14] another such example was constructed by defining $u_1 = -\frac{2}{3}((y_3)^2 + 4(y_4)^2 + (y_5)^2)$, $u_2 = \frac{2\sqrt{3}}{3}((y_3)^2 - (y_5)^2)$, $u_3 = \frac{2}{3}(y_1y_3 - \sqrt{3}y_2y_3 - 3y_4y_5 - (y_5)^2)$, $u_4 = \frac{8}{3}y_1y_4$ and $u_5 = \frac{2}{3}(y_1y_5 + \sqrt{3}y_2y_5 + 3y_3y_4 + y_3y_5)$. These examples also have $\mathfrak{so}(3) \subset \mathfrak{so}(5)$ as screen holonomy. We do not know whether the three examples are locally isometric.

On the other hand one can construct a manifold with the same holonomy but with different geometric properties, i.e. which does not have light-like hypersurface curvature, by the following construction. Let g be the Riemannian metric on $Sl(3, \mathbb{R})/SO(3, \mathbb{R})$ and consider the Lorentzian manifold

$$(M := \mathbb{R}^2 \times (Sl(3, \mathbb{R})/SO(3, \mathbb{R})), h := 2\mathrm{d}x\mathrm{d}z + f\mathrm{d}z^2 + g.)$$

If f is sufficiently general this manifold is indecomposable and has holonomy algebra $\mathfrak{so}(3) \ltimes \mathbb{R}^5$ or $(\mathbb{R} \oplus \mathfrak{so}(3)) \ltimes \mathbb{R}^5$. But, its curvature restricted to $\left(\frac{\partial}{\partial x}\right)^\perp$ does not vanish because $\mathcal{R}\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l}\right)$ equals the curvature of $Sl(3, \mathbb{R})/SO(3, \mathbb{R})$.

This example as well as the curvature calculations in local coordinates show that the $\mathfrak{so}(n)$-part of the holonomy of a Lorentzian manifold with light-like hypersurface curvature is not necessarily trivial. Due to a recent result in [14] one can even show that any possible screen holonomy, i.e. any Riemannian holonomy group, can occur as a screen holonomy of a Lorentzian manifold with light-like hypersurface curvature — although the defining condition on the curvature of such a manifold is quite strong. We will now indicate why this is the case.

The classification of possible screen holonomies was based on the notion of *weak curvature endomorphisms* and *weak-Berger algebras* which we have introduced in [17]. Weak curvature endomorphisms are defined for Lie algebras $\mathfrak{g} \subset \mathfrak{so}(n)$ by a Bianchi-identity:

$$\mathcal{B}(\mathfrak{g}) := \{Q \in Hom(\mathbb{R}^n, \mathfrak{g}) \mid \langle Q(x)y, z\rangle + \langle Q(y)z, x\rangle + \langle Q(z)x, y\rangle\}.$$

$\mathcal{B}(\mathfrak{g})$ is the kernel of the homomorphism $\lambda : Hom(\mathbb{R}^n, \mathfrak{g}) \to \Lambda^3(\mathbb{R}^n)^*$ which is the combination of skew symmetrisation and dualisation by means of the scalar product $\langle ., .\rangle$. $\mathfrak{g}$ is called *weak-Berger algebra* if and only if

$$\mathfrak{g} = \mathsf{span}\{Q(x) \mid Q \in \mathcal{B}(\mathfrak{g}), x \in \mathbb{R}^n\}.$$

In [19] and [20] we showed that any weak-Berger algebra is a Riemannian holonomy algebra. On the other hand, any Riemannian holonomy algebra can be realised as a screen holonomy of a Lorentzian manifold with recurrent or parallel light-like vector field by the following construction. Let (N, g) be a Riemannian manifold and $f \in C^\infty(N \times \mathbb{R}^2)$ a smooth function which is sufficiently generic. Then $M := \mathbb{R}^2 \times N$ with the metric

$$h := 2\mathrm{d}x\mathrm{d}z + f\mathrm{d}z^2 + g$$

is a Lorentzian manifold with recurrent light-like vector field and the screen holonomy of (M, h) is equal to the Riemannian holonomy $Hol_p(N, g)$ (see [18] for details). But it was an open

question whether for any of the four types of holonomy in [5] any Riemannian holonomy can be realised as screen holonomy. In [14] it was shown that this is possible. We will now describe briefly parts of this method which we will need to construct further examples. This construction uses the fact that the screen holonomy $\mathfrak{g}$ is a weak-Berger algebra. For details of the following see [14].

First, for a weak-Berger algebra $\mathfrak{g} \subset \mathfrak{so}(n)$ one fixes weak curvature endomorphisms $Q_A \in \mathcal{B}(\mathfrak{g})$ for $A = 1, \ldots, N$ and a basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$, orthonormal w.r.t. $\langle ., . \rangle$. Now one defines the following polynomials on $\mathbb{R}^{n+1}$,

$$u_i(y_1, \ldots, y_n, z) := \sum_{A=1}^{N} \sum_{k,l=1}^{n} \frac{(A-1)!}{3} \langle Q_A(e_k)e_l + Q_A(e_l)e_k, e_i \rangle y_k y_l z^A, \tag{15}$$

and the following Lorentzian metric on $\mathbb{R}^{n+2}$,

$$h = 2\mathrm{d}x\,\mathrm{d}z + f\mathrm{d}z^2 + 2\sum_{i=1}^{n} u_i \mathrm{d}y_i \mathrm{d}z + \sum_{k=1}^{n} \mathrm{d}y_k^2, \tag{16}$$

where f is a function on $\mathbb{R}^{n+2}$. This metric is analytic, hence its holonomy is generated by the derivations of the curvature tensor. But the metric is constructed in such a way that the only non-vanishing $\mathfrak{so}(n)$-parts of the curvature and its derivatives satisfy

$$pr_{\mathfrak{so}(n)} \left[(\underbrace{\nabla_{\frac{\partial}{\partial z}} \ldots \nabla_{\frac{\partial}{\partial z}}}_{(A-1)\text{-times}} \mathcal{R}) \left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial z} \right) \right] = Q_A(e_i), \tag{17}$$

for $A = 1, \ldots, N$ and $i = 1, \ldots, n$. If one now starts this construction with $Q_1, \ldots, Q_N$ spanning $\mathcal{B}(\mathfrak{g})$, e.g. a basis of $\mathcal{B}(\mathfrak{g})$, then the derivatives of the curvature will span $\mathfrak{g}$. Hence the weak-Berger algebra $\mathfrak{g}$ we started with is the screen holonomy of $(\mathbb{R}^{n+2}, h)$. But, more importantly, in [14] it is proven that, for any of the four types of indecomposable, non-irreducible Lorentzian holonomy in [5] the function f can be chosen in such a way that the holonomy of h belongs to this type.

For our purposes it is important that the constructed metric h admits light-like hypersurface curvature due to the description in coordinates in Proposition 9. We obtain the following result.

Proposition 12. *For any of the four types of indecomposable, non-irreducible Lorentzian holonomy and any Riemannian holonomy algebra $\mathfrak{g}$ there is a Lorentzian manifold (M, h) with light-like hypersurface curvature such that the holonomy of (M, h) is of the given type and its screen holonomy is equal to $\mathfrak{g}$.*

This result is most remarkable as the curvature conditions on a manifold with light-like hypersurface curvature are very strong and only a slight generalisation of the curvature conditions imposed on a pp-wave, which has trivial screen holonomy.

The method described above gives a construction principle for Lorentzian manifolds with light-like hypersurface curvature under the assumption that the weak curvature endomorphism are known. But since every weak-Berger algebra $\mathfrak{g}$ is a Riemannian holonomy algebra and thus a Berger algebra, i.e.

$$\mathfrak{g} = \mathsf{span}\{R(x, y) \mid R \in \mathcal{K}(\mathfrak{g}), x, y \in \mathbb{R}^n\}$$

where $\mathcal{K}(\mathfrak{g})$ are the following curvature endomorphisms

$$\mathcal{K}(\mathfrak{g}) = \{R \in Hom(\Lambda^2 \mathbb{R}^n, \mathfrak{g}) \mid R(x, y)z + R(y, z)x + R(z, x)y = 0\},$$

sometimes it is sufficient to know the space $\mathcal{K}(\mathfrak{g})$. We will illustrate this in the following construction which generalises Example 1. First we note that both spaces of curvature endomorphisms, $\mathcal{B}(\mathfrak{g})$ and $\mathcal{K}(\mathfrak{g})$, are $\mathfrak{g}$-modules related in the following way.

Lemma 7. *Let* $\mathfrak{g} \subset \mathfrak{so}(n)$. *Then the vector space*

$$\mathcal{R}(\mathfrak{g}) := \mathsf{span}\{R(x, .) \in Hom(\mathbb{R}^n, \mathfrak{g}) \mid R \in \mathcal{K}(\mathfrak{g}), \ x \in \mathbb{R}^n\}$$

is a $\mathfrak{g}$-*submodule of* $\mathcal{B}(\mathfrak{g})$.

Proof. Because of the defining Bianchi-identities, $\mathcal{R}(\mathfrak{g}) \subset \mathcal{B}(\mathfrak{g})$ is ensured. For $R \in \mathcal{K}(\mathfrak{g})$ we have

$$(A \cdot R(x, .))(y) = [A, R(x, y)] - R(x, Ay) = (A \cdot R)(x, y) + R(Ax, y) \in \mathcal{R}(\mathfrak{g}),$$

which shows that $\mathcal{R}(\mathfrak{g})$ is also a submodule.   □

This lemma shows that a priori any Berger algebra is a weak-Berger algebra whereas the other implication requires a proof based on representation theory (see [17,19] and [20]). Nevertheless, we can apply the lemma in order to construct examples of Lorentzian manifolds with light-like hypersurface curvature and the holonomy of a Riemannian symmetric space G/K as screen holonomy.

Suppose G/K is a Riemannian symmetric space of semisimple type and of dimension n. In particular, the Lie algebras of G and K satisfy $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ with $\mathfrak{k}$ a subalgebra acting irreducibly on $\mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$. The metric on G/K corresponds to an invariant inner product $\langle ., . \rangle$ which is a multiple of the Killing form B of $\mathfrak{g}$. The holonomy group of G/K is K acting by the adjoint representation on $\mathfrak{m} \simeq T_{[e]}(G/K)$. Suppose $X_1, \ldots, X_n$ is a basis of $\mathfrak{m}$ which is orthogonal with respect to B. Using these ingredients we define the following polynomials on $\mathbb{R}^{n+1}$:

$$u_i^{(G,K)}(y_1, \ldots, y_n, z)$$
$$:= \sum_{j,k,l=1}^n \frac{(j-1)!}{3}(B([X_j, X_k], [X_l, X_i]) + B([X_j, X_l], [X_k, X_i]))y_k y_l z^j,$$

where $[., .]$ is the commutator in $\mathfrak{g}$ and B the Killing form. Again we define a Lorentzian metric on $\mathbb{R}^{n+2}$ by

$$h^{(G,K)} = 2\mathrm{d}x\,\mathrm{d}z + f\,\mathrm{d}z^2 + 2\sum_{i=1}^n u_i^{(G,K)}\mathrm{d}y_i\,\mathrm{d}z + \sum_{k=1}^n \mathrm{d}y_k^2,$$

for f being a smooth function on $\mathbb{R}^{n+2}$. In this situation the following proposition holds.

Proposition 13. *Let* G/K *be an irreducible Riemannian symmetric space of dimension n. Then the Lorentzian metric* $h^{(G,K)}$ *on* $\mathbb{R}^{n+2}$ *has light-like hypersurface curvature and its screen holonomy is equal to the holonomy of the Riemannian symmetric space* G/K, *i.e. is equal to* K.

Proof. The proof relies on the method described above. Since G/K is a Riemannian symmetric space, the curvature endomorphisms of $\mathfrak{k}$ satisfy $\mathcal{K}(\mathfrak{k}) = \mathbb{R} \cdot [., .]$, where $[., .]$ is the commutator of $\mathfrak{g}$. Since $\mathfrak{k}$ is the holonomy algebra of this space we get $\mathfrak{k} = \mathsf{span}\{[X, Y] \mid X, Y \in \mathfrak{m}\}$. Hence

for a basis $X_1, \ldots, X_n$ of $\mathfrak{m}$, the $Q_j := [X_j, \cdot]$ are spanning the submodule $\mathcal{R}(\mathfrak{k})$ in $\mathcal{B}(\mathfrak{k})$ by Lemma 7 and generate the whole Lie algebra $\mathfrak{k}$. In this situation, if the basis X_i is assumed to be orthogonal w.r.t. the Killing form B, we obtain for the terms in (15)

$$
\begin{aligned}
B(Q_j(X_k)X_l + Q_j(X_l)X_k, X_i) &= B([X_j, X_k]X_l + [X_j, X_l]X_k, X_i) \\
&= B([[X_j, X_k], X_l] + [[X_j, X_l], X_k], X_i) \\
&= B([X_j, X_k], [X_l, X_i]) + B([X_j, X_l], [X_k, X_i]).
\end{aligned}
$$

Hence, the curvature of $h^{(G,K)}$ satisfies (17) which implies that the holonomy of $h^{(G,H)}$ is equal to K. $\quad\square$

Again, as in Example 1, a Lorentzian manifold with the same screen holonomy can be obtained using the metric $h = 2\mathrm{d}x\mathrm{d}z + f\mathrm{d}z^2 + g$ where g is the Riemannian metric of G/K. But this manifold does not have light-like hypersurface curvature and is therefore not locally isometric to $h^{(G,K)}$.

In principle, the method of [14] works for any Riemannian holonomy algebra, also non-symmetric ones, if one is able to calculate $\mathcal{B}(\mathfrak{g})$. As in Proposition 13 one could also try to use the submodule $\mathcal{R}(\mathfrak{g})$, but for non-symmetric Riemannian holonomy algebras, $\mathcal{K}(\mathfrak{g})$ can be very big and thus the calculations complicated. Another way is to use other, easier submodules of $\mathcal{B}(\mathfrak{g})$. This methods works if $\mathfrak{g}$ is simple, since any submodule of $\mathcal{B}(\mathfrak{g})$ generates a non-trivial ideal in $\mathfrak{g}$ which has to be equal to $\mathfrak{g}$ for $\mathfrak{g}$ simple. For example, in the case of the exceptional Lie algebra $\mathfrak{g}_2 \subset \mathfrak{so}(V)$ with $V = \mathbb{R}^7$ the $\mathfrak{g}_2$-module $Hom(V, \mathfrak{g}_2)$ which contains $\mathcal{B}(\mathfrak{g}_2)$ splits into the direct sum of $V_{[1,1]}$, $\odot_0^2 V^*$ and V, where $V_{[1,1]}$ is the 64-dimensional $\mathfrak{g}_2$-module of highest weight $(1, 1)$, and $\odot_0^2 V^*$ is the 27-dimensional module of highest weight $(2, 0)$. Since $\mathcal{B}(\mathfrak{g}_2)$ is the kernel of the skew symmetrisation

$$
\begin{array}{ccc}
\lambda \quad : & Hom(V, \mathfrak{g}_2) & \longrightarrow & \Lambda^3 V^* \\
& // & & \backslash\backslash \\
V_{[1,1]} \oplus \odot_0^2 V^* \oplus V & & \odot_0^2 V^* \oplus V \oplus \mathbb{C}
\end{array}
$$

a dimension analysis shows that $\mathcal{B}(\mathfrak{g})$ must contain $V_{[1,1]}$. Thus, by choosing a basis of $V_{[1,1]}$ a metric of the form (16) with coefficients as in (15) can be defined and one obtains a Lorentzian manifold with light-like hypersurface curvature and screen holonomy G_2.

Acknowledgement

References

[1] C. Atindogbe, K.L. Duggal, Conformal screen on lightlike hypersurfaces, Int. J. Pure Appl. Math. 11 (4) (2004) 421–442.
[2] H. Baum, Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten, in: Teubner-Texte zur Mathematik, vol. 41, Teubner-Verlagsgesellschaft, 1981.
[3] H. Baum, Conformal Killing spinors and special geometric structures in Lorentzian geometry — a survey, in: Proceedings of the Workshop on Special Geometric Structures in String Theory, Bonn, September 2001. Proceedings archive of the EMS Electronic Library of Mathematics, www.univie.ac.at/EMIS/proceedings/, 2002.
[4] A. Bejancu, K.L. Duggal, Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, in: Mathematics and Its Applications, vol. 364, Kluwer Academic Press, 1996.

[5] L. Bérard-Bergery, A. Ikemakhen, On the holonomy of Lorentzian manifolds, in: Differential Geometry: Geometry in Mathematical Physics and Related Topics, Los Angeles, CA, 1990, in: Proc. Sympos. Pure Math., vol. 54, Amer. Math. Soc., Providence, RI, 1993, pp. 27–40.

[6] M.M. Berger, Sur les groupes d'holonomie homogène des variétés a connexion affine et des variétés riemanniennes, Bull. Soc. Math. France 83 (1955) 279–330.

[7] N. Bezvitnaya, Lightlike foliations on Lorentzian manifolds with weakly irreducible holonomy algebra, math.DG/0506101 at http://arxiv.org, 2005.

[8] M. Blau, J. Figueroa-O'Farrill, C. Hull, G. Papadopoulos, Penrose limits and maximal supersymmetry, Classical Quantum Gravity 19 (10) (2002) L87–L95.

[9] C. Boubel, Sur l'holonomie des variétés pseudo-riemanniennes Ph.D. Thesis, Université Henri Poincaré, Nancy, 2000.

[10] H.W. Brinkmann, Einstein spaces which are mapped conformally on each other, Math. Ann. 94 (1925) 119–145.

[11] M. Cahen, N. Wallach, Lorentzian symmetric spaces, Bull. Amer. Math. Soc. 79 (1970) 585–591.

[12] G. de Rham, Sur la réducibilité d'un espace de Riemann, Math. Helv. 26 (1952) 328–344.

[13] K.L. Duggal, A. Giménez, Lightlike hypersurfaces of Lorentzian manifolds with distinguished screen, J. Geom. Phys. 55 (1) (2005) 107–122.

[14] A.S. Galaev, Metrics that realize all types of Lorentzian holonomy algebras, math.DG/0502575 at http://arxiv.org, 2005.

[15] A. Ikemakhen, Examples of indecomposable non-irreducible Lorentzian manifolds, Ann. Sci. Math. Québec 20 (1) (1996) 53–66.

[16] I. Kath, Killing Spinors on Pseudo-Riemannian Manifolds, Habilitationsschrift, Humboldt-Universität Berlin, 1999.

[17] T. Leistner, Berger algebras, weak-Berger algebras and Lorentzian holonomy, SFB 288-Preprint no. 567, ftp://ftp-sfb288.math.tu-berlin.de/pub/Preprints/preprint567.ps.gz, 2002.

[18] T. Leistner, Lorentzian manifolds with special holonomy and parallel spinors, in: The proceedings of the 21st winter school "Geometry and Physics", 13–20 January, Srni, 2001, in: Supplemento ai Rendiconti del Circolo Matematico di Palermo. Serie II, vol. 69, 2002, pp. 131–159.

[19] T. Leistner, Towards a classification of Lorentzian holonomy groups, math.DG/0305139 at http://arxiv.org, 2003.

[20] T. Leistner, Towards a classification of Lorentzian holonomy groups. Part II: Semisimple, non-simple weak Berger algebras, math.DG/0309274 at http://arxiv.org, 2003.

[21] T. Leistner, Holonomy and Parallel Spinors in Lorentzian Geometry, Logos Verlag, 2004. Dissertation, Mathematisches Institut der Humboldt-Universität Berlin, 2003.

[22] T. Leistner, Conformal holonomy of C-spaces, Ricci-flat, and Lorentzian manifolds, Differential Geom. Appl. (2005) (in press). math.DG/0501239 at http://arxiv.org.

[23] R. Schimming, Riemannsche Räume mit ebenfrontiger und mit ebener Symmetrie, Math. Nachr. 59 (1974) 128–162.

[24] A. G. Walker, On parallel fields of partially null vector spaces, Quart. J. Math. 20 (1949) 135–145.

[25] H. Wu, On the de Rham decomposition theorem, Illinois J. Math. 8 (1964) 291–311.